[Using (B4) and $\gamma_\rho^2=2$ we find for D_{13}, $\delta=0.38°$, 0.51°, 0.86°, 1.31° at 300, 400, 500, and 600 MeV. For D_{15} one finds 0.22°, 0.25°, 0.4°, 0.6°, 0.84°, 1.1°, 1.37° at 300, 400, 500, 600, 700, 800, 900 MeV. For F_{15} δ is 0.36°, 0.50° and 0.68° at 700, 800, and 900 MeV. For H_{19} δ is 0.37°, 0.56° and 1.05° at 1700, 2000, and 2500 MeV. The same parameters give for P_{11} $\delta=12°$ at 900 MeV.]

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Effect of the Baryon Excited States on the $N-\Lambda$ and $\Lambda-\Lambda$ Forces

Y. NOGAMI*† AND F. J. BLOORE*‡
Division of Pure Physics, National Research Council, Ottawa, Ontario, Canada
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The N-Λ and Λ-Λ potentials caused by the exchange of two pions are calculated in the static theory, taking into account the resonance Y_1^* in the π–Λ system and the (3–3) resonance in the π–N system. The recoil of the baryons is included in an approximate way. It is shown that the presence of these resonances diminishes the spin-dependent part of the central potential and the tensor potential, and increases the spin-independent part of the central potential. The triplet potential turns out to be slightly stronger than the singlet potential at large distances, and slightly weaker than it closer in. If the resonances are omitted, the triplet potential is the stronger over the whole range. This last result is in mild disagreement with other work. Its relation to the choice of a one-channel or two-channel formalism is discussed.

1. INTRODUCTION

SOME experimental evidence on hypernuclei and on double-hypernuclei is now available and some phenomenological analyses of this evidence have been made with a view to determining the nature of the $N-\Lambda$[1] and $\Lambda-\Lambda$ forces.[2]

Various workers[3] have estimated the two-pion exchange contributions to these potentials using meson theory. However, no account seems yet to have been given of the effect upon these forces produced by the Y_1^* resonance in the $\pi-\Lambda$ system and the $3-3$ resonance in the $\pi-N$ system together.[4] The main purpose of this paper is to estimate this effect.

We shall take the $\Sigma-\Lambda$ parity to be even, as has now been almost conclusively established,[5] and we shall make the experimentally probable assumption that the Y_1^* resonance at 1385 MeV in the $\pi-\Lambda$ system is a $P_{3/2}$ state,[6] having the same mechanism as the $I=J=\frac{3}{2}$ resonance in the $\pi-N$ system. The Chew-Low theory for the pion-nucleon interaction can then be extended in a straightforward way to the pion-hyperon interaction, and the $N-\Lambda$ and $\Lambda-\Lambda$ potentials can be calculated by the method given by Miyazawa,[7] a method in which the resonances of the $\pi-N$ and $\pi-\Lambda$ systems can be treated.

It has been pointed out by Charap and Fubini and by Gupta[8] that the static limit of the two-pion exchange potential is not well defined. The difficulty comes from the fact that, when the two-pion exchange potential $V(x)$ is written in the form

$$V(x)=\int_{(2m_\pi)^2}^{\infty} dm^2\rho(m^2)\exp(-mx)/x, \qquad (1.1)$$

the inverse baryon mass expansion of the spectral function $\rho(m^2)$ does not converge at the lower mass end ($m\rightarrow 2m_\pi$). The relativistic effect is therefore important in the asymptotic region ($x\rightarrow\infty$), where the static limit would appear to be most justified. Akiba[9] has examined the accuracy of the inverse nucleon mass expansion, showing that this expansion provides us with a reasonable numerical approximation. Our calculation will be meaningful except for extremely large distances where $|V(x)|$ will be negligibly small, and of course for very short distances.

* National Research Council Postdoctorate Fellow.
† Present address: Department of Physics, Battersea College of Technology, London, England.
‡ Present address: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England.
[1] See R. H. Dalitz, Enrico Fermi Institute for Nuclear Studies Report No. EFINS-62-9, 1962 (unpublished) for a review of the $N-\Lambda$ interaction.
[2] H. Nakamura, Progr. Theoret. Phys. (Kyoto) **30**, 84 (1963); S. Iwao, Nucl. Phys. **26**, 1 (1962); R. H. Dalitz, Phys. Letters **5**, 53 (1963).
[3] J. J. de Swart and C. K. Iddings, Phys. Rev. **126**, 2810 (1962) and references cited therein; J. J. de Swart, Phys. Letters **5**, 58 (1963); A. Deloff, *ibid.* **5**, 147 (1963); R. Schrils and B. W. Downs, Phys. Rev. **131**, 390 (1963).
[4] M. Uehara [Progr. Theoret. Phys. (Kyoto) **24**, 629 (1960)] has discussed the effect of the (3–3) resonance upon the $N-\Lambda$ interaction.
[5] R. D. Tripp, M. B. Watson, and M. Ferro-Luzzi, Phys. Letters **8**, 175 (1962); H. Courant, H. Filthuth, P. Franzini, R. G. Glasser, *et al.*, Phys. Rev. Letters **10**, 409 (1963); R. H. Capps, Nuovo Cimento **26**, 1339 (1962).
[6] L. Bertanza, V. Brisson, P. L. Connolly, E. L. Hart, I. S. Mittra, *et al.*, Phys. Rev. Letters **10**, 176 (1963); J. B. Shafer, J. Murray, and D. O. Huwe, Phys. Rev. Letters **10**, 179 (1963).
[7] H. Miyazawa, Phys. Rev. **104**, 1741 (1956); M. Konuma, H. Miyazawa, and S. Otsuki, Progr. Theoret. Phys. (Kyoto) **19**, 17 (1958).
[8] J. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959); **15**, 73 (1960); J. M. Charap and M. J. Tausner, Nuovo Cimento **18**, 316 (1960); S. N. Gupta, Phys. Rev. **117**, 1146 (1960).
[9] T. Akiba, Progr. Theoret. Phys. (Kyoto) **27**, 241 (1962).

There is an arbitrariness in the definitions of the $N-\Lambda$ and $\Lambda-\Lambda$ potentials which is connected with the number of channels which are to be considered in the solution of the Schrödinger equation. If we took two channels, $N-\Lambda$ and $N-\Sigma$, then the potential V would be a 2×2 matrix,

$$V=\begin{bmatrix} V_{\Lambda\Lambda} & V_{\Lambda\Sigma} \\ V_{\Sigma\Lambda} & V_{\Sigma\Sigma} \end{bmatrix}. \tag{1.2}$$

We shall, however, work in a one-channel formalism, so that V is now a single element.

The main difference between the one-channel and the two-channel formalisms lies in the treatment of the diagram in Fig. 1 which possesses an intermediate state having no virtual meson. (We shall discuss only the $N-\Lambda$ case as the $\Lambda-\Lambda$ case is quite similar.) In the two-channel formalism, Lichtenberg and Ross,[10] following the treatment of the nucleon-nucleon interaction given by Brueckner and Watson,[11] have argued that Fig. 1 is a repetition of the simplest graph for $V_{\Lambda\Sigma}$ and must be omitted from the calculation of $V_{\Lambda\Lambda}$. In the one-channel approach, however, Fig. 1 must be included as it cannot now be interpreted as a repetition of lower order graphs. In the two-channel formalism there are other interpretations of Fig. 1 which are not equivalent to that of Lichtenberg and Ross, and which in fact lead to different $N-\Lambda$ potentials. For example, if we adopt the prescription of perturbation theory which leads to the *TMO* potential in the $N-N$ case,[12] we should obtain a nonzero contribution from Fig. 1 corresponding to the "probability term" of Fukuda, Sawada, and Taketani.[13] This ambiguity is absent from the one-channel treatment, which is an important consideration in its favor.

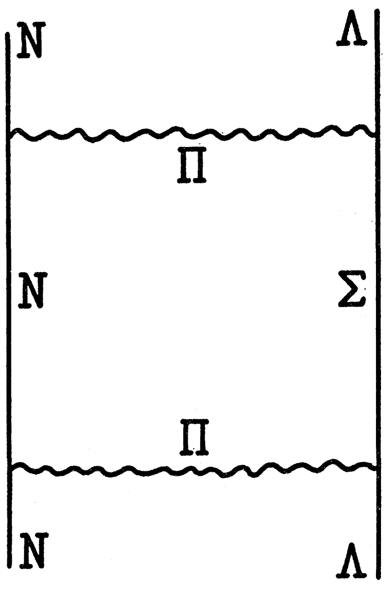


FIG. 1. Repetition diagram.

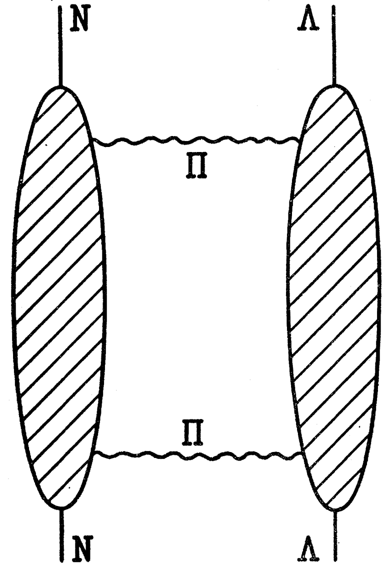


FIG. 2. Diagram for the $N-\Lambda$ interaction.

Unfortunately, as noted by Lichtenberg and Ross,[10] the one-channel formalism possesses a severe disadvantage which is also connected with Fig. 1. The mathematical expression corresponding to Fig. 1 contains an integrand whose denominator involves the factor $(\Delta+T)$ where Δ is the mass difference $M_\Sigma-M_\Lambda$ and T is the kinetic energy of the baryons in the intermediate state. In the static model, the baryons are considered to be fixed and T is ignored. The validity of this approximation is discussed and an attempt is made in Sec. 4 to take account of the motion of the baryons in an approximate way.

In Secs. 2 and 3 the potentials for the $N-\Lambda$ and $\Lambda-\Lambda$ interactions are derived. The calculations of the potentials are presented in Sec. 5 and are discussed in the light of other work in Sec. 6. The range of validity of a useful approximation suggested by Matsumoto, Hamada, and Sugawara[14] is discussed in an Appendix.

2. THE $N-\Lambda$ POTENTIAL

We shall use units in which $\hbar=c=m_\pi=1$.

Following Miyazawa,[7] we write the S-matrix element corresponding to the lowest order diagram Fig. 2 as

$$S=-\frac{1}{2}\frac{1}{(2\pi)^8}\sum_{i,i'}\int\int d^4k d^4k' \frac{\langle i'k'|S^{(N)}|ik\rangle\langle i'-k'|S^{(\Lambda)}|i-k\rangle}{(k_0{}^2-\omega_k{}^2)(k_0'^2-\omega_{k'}{}^2)}, \tag{2.1}$$

[10] D. B. Lichtenberg and M. H. Ross, Phys. Rev. **107**, 1714 (1957).
[11] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).
[12] M. Taketani, S. Machida, and S. Onuma, Progr. Theoret. Phys. (Kyoto) **7**, 45 (1952).
[13] N. Fukuda, K. Sawada, and M. Taketani, Progr. Theoret. Phys. (Kyoto) **12**, 156 (1954).
[14] T. Matsumoto, T. Hamada, and M. Sugawara, Progr. Theoret. Phys. (Kyoto) **10**, 199 (1953).

where

$$\langle i'k'|S^{(N)}|ik\rangle=2\pi i\delta(k_0-k_0')[A_N(k_0)\tau_i\tau_{i'}(\boldsymbol{\sigma}_N\cdot\mathbf{k})(\boldsymbol{\sigma}_N\cdot\mathbf{k}')+B_N(k_0)\{\tau_i\tau_{i'}(\boldsymbol{\sigma}_N\cdot\mathbf{k}')(\boldsymbol{\sigma}_N\cdot\mathbf{k}) +\tau_{i'}\tau_i(\boldsymbol{\sigma}_N\cdot\mathbf{k})(\boldsymbol{\sigma}_N\cdot\mathbf{k}')\}+C_N(k_0)\tau_{i'}\tau_i(\boldsymbol{\sigma}_N\cdot\mathbf{k}')(\boldsymbol{\sigma}_N\cdot\mathbf{k})]\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}_N]v_kv_{k'},\quad(2.2)$$

$$\langle i'k'|S^{(\Lambda)}|ik\rangle=2\pi i\delta_{ii'}\delta(k_0-k_0')[A_\Lambda(k_0)(\boldsymbol{\sigma}_\Lambda\cdot\mathbf{k})(\boldsymbol{\sigma}_\Lambda\cdot\mathbf{k}')+C_\Lambda(k_0)(\boldsymbol{\sigma}_\Lambda\cdot\mathbf{k}')(\boldsymbol{\sigma}_\Lambda\cdot\mathbf{k})]\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}_\Lambda]v_kv_{k'}\quad(2.3)$$

are the matrix elements for the $\pi-N$ and $\pi-\Lambda$ scattering parts. The function v_k is a cutoff factor which we choose to be $\exp(-k^2/2k_m^2)$, where k_m is the momentum corresponding to the nucleon mass.[15] The functions A_N, B_N, C_N, A_Λ, and C_Λ are given in terms of the $\pi-N$ and $\pi-\Lambda$ scattering cross sections by the dispersion relations in the static approximation,

$$A_N(k_0)=\frac{4\pi f_N^2}{k_0-i\epsilon}+\frac{1}{4\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}(p)}{\omega_p-k_0-i\epsilon}\right)+\frac{1}{36\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{4\sigma_{11}(p)+4\sigma_{13}(p)+\sigma_{33}(p)}{\omega_p+k_0-i\epsilon}\right),\quad(2.4a)$$

$$B_N(k_0)=\frac{1}{12\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}(p)+2\sigma_{13}(p)}{\omega_p-k_0-i\epsilon}\right)+\frac{1}{12\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}(p)+2\sigma_{13}(p)}{\omega_p+k_0-i\epsilon}\right),\quad(2.4b)$$

$$C_N(k_0)=\frac{-4\pi f_N^2}{k_0+i\epsilon}+\frac{1}{36\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{4\sigma_{11}(p)+4\sigma_{13}(p)+\sigma_{33}(p)}{\omega_p-k_0-i\epsilon}\right)+\frac{1}{4\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}(p)}{\omega_p+k_0-i\epsilon}\right),\quad(2.4c)$$

$$A_\Lambda(k_0)=\frac{4\pi f_\Lambda^2}{\Delta+k_0-i\epsilon}+\frac{1}{2\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_3(p)}{\omega_p-k_0-i\epsilon}\right)+\frac{1}{6\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{2\sigma_1(p)+\sigma_3(p)}{\omega_p+k_0-i\epsilon}\right),\quad(2.5a)$$

$$C_\Lambda(k_0)=\frac{4\pi f_\Lambda^2}{\Delta-k_0-i\epsilon}+\frac{1}{6\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{2\sigma_1(p)+\sigma_3(p)}{\omega_p-k_0-i\epsilon}\right)+\frac{1}{2\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_3(p)}{\omega_p+k_0-i\epsilon}\right).\quad(2.5b)$$

Here $f_N^2(=0.08)$ and f_Λ^2 are the renormalized $\pi-N$ and $\pi-\Lambda-\Sigma$ coupling constants, $\sigma_{2I,2J}$ is the total cross section of the $\pi-N$ scattering in the state (I,J) and σ_{2J} is the total cross section of the $\pi-\Lambda$ scattering in the state with angular momentum $J(=\frac{1}{2}$ or $\frac{3}{2})$.[16]

If we insert (2.2) and (2.3) into (2.1) we obtain S in the form $S=-2\pi i\delta(0)V(|\mathbf{x}_N-\mathbf{x}_\Lambda|)$. We can interpret V as the potential acting between the two baryons at $\mathbf{x}_N$ and $\mathbf{x}_\Lambda$. On performing the integrations over k_0, k_0' and over the directions of $\mathbf{k}$ and $\mathbf{k}'$ we find

$$V(x)=-[(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla}_z)(\boldsymbol{\sigma}_\Lambda\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\sigma}_\Lambda\cdot\boldsymbol{\nabla}_z)\Xi(y,z)+(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla}_z)(\boldsymbol{\sigma}_\Lambda\cdot\boldsymbol{\nabla}_z)(\boldsymbol{\sigma}_\Lambda\cdot\boldsymbol{\nabla}_y)Z(y,z)]_{y=z=x},\quad(2.6)$$

where

$$\Xi(y,z)=3(4\pi)^2\Big[f_N^2f_\Lambda^2\{F_{0\Delta}(y,z)+2G_{0\Delta}(y,z)\}+\frac{f_N^2}{12\pi^2}\int_0^\infty\frac{dp}{\omega_p}\sigma_3(p)\{2F_{0\omega_p}(y,z)+G_{0\omega_p}(y,z)\}$$
$$+\frac{f_\Lambda^2}{18\pi^2}\int_0^\infty\frac{dp}{\omega_p}\sigma_{33}(p)\{2F_{\Delta\omega_p}(y,z)+G_{\Delta\omega_p}(y,z)\}$$
$$+\frac{1}{216\pi^4}\int_0^\infty\int_0^\infty\frac{dpdp'}{\omega_p\omega_{p'}}\sigma_{33}(p)\sigma_3(p')\{4F_{\omega_p\omega_{p'}}(y,z)+5G_{\omega_p\omega_{p'}}(y,z)\}\Big],\quad(2.7a)$$

$$Z(y,z)=3(4\pi)^2\Big[f_N^2f_\Lambda^2F_{0\Delta}(y,z)+\frac{f_N^2}{12\pi^2}\int_0^\infty\frac{dp}{\omega_p}\sigma_3(p)\{2F_{0\omega_p}(y,z)+3G_{0\omega_p}(y,z)\}$$
$$+\frac{f_\Lambda^2}{18\pi^2}\int_0^\infty\frac{dp}{\omega_p}\sigma_{33}(p)\{2F_{\Delta\omega_p}(y,z)+3G_{\Delta\omega_p}(y,z)\}$$
$$+\frac{1}{216\pi^4}\int_0^\infty\int_0^\infty\frac{dpdp'}{\omega_p\omega_{p'}}\sigma_{33}(p)\sigma_3(p')\{4F_{\omega_p\omega_{p'}}(y,z)+3G_{\omega_p\omega_{p'}}(y,z)\}\Big].\quad(2.7b)$$

[15] No cutoff factor appears in Miyazawa's formula, but we introduce a cutoff factor, following G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956), for the convenience of computation.

[16] Cutoff factors in the integrands in (2.4) and (2.5) are suppressed. For the Low equations of the pion-hyperon scatterings, see, e.g., A. Komatsuzawa, R. Sugano, and Y. Nogami, Progr. Theoret. Phys. (Kyoto) **21**, 151 (1959), and R. H. Capps and M. Nauenberg, Phys. Rev. **118**, 593 (1959).

The functions F and G are defined as

$$F_{\lambda\mu}(y,z)=\frac{1}{2(2\pi)^3 yz}\int_0^\infty dk\frac{k\sin k(y+z)}{\omega_k(\omega_k+\lambda)(\omega_k+\mu)}v_k^4, \tag{2.8a}$$

$$G_{\lambda\mu}(y,z)=\frac{1}{2(2\pi)^3 yz(\lambda+\mu)}\int_0^\infty dk\frac{k\sin k(y+z)}{(\omega_k+\lambda)(\omega_k+\mu)}v_k^4. \tag{2.8b}$$

In writing (2.7) we have omitted terms which contain $\sigma_{11}(p)$, $\sigma_{13}(p)$ or $\sigma_1(p)$, as we assume that the $\pi-N$ scattering is dominated by the $3-3$ resonance and that the $\pi-\Lambda$ scattering is dominated by the Y_1^* resonance in $\sigma_3(p)$.

The potential (2.6) can be expressed as a sum of central, spin-dependent and tensor parts

$$V(x)=V_0(x)+V_S(x)(\boldsymbol{\sigma}_N\cdot\boldsymbol{\sigma}_\Lambda)+V_T(x)S_{N\Lambda}, \tag{2.9}$$

where

$$S_{N\Lambda}=3\frac{(\boldsymbol{\sigma}_N\cdot\mathbf{x})(\boldsymbol{\sigma}_\Lambda\cdot\mathbf{x})}{x^2}-(\boldsymbol{\sigma}_N\cdot\boldsymbol{\sigma}_\Lambda)$$

and

$$V_0(x)=-\left[\frac{2}{x^2}\frac{\partial^2}{\partial y\partial z}+\frac{\partial^4}{\partial y^2\partial z^2}\right][\Xi(y,z)+Z(y,z)]_{y=z=x}, \tag{2.10a}$$

$$V_S(x)=\frac{2}{3x}\left[\frac{1}{x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\frac{\partial^2}{\partial y\partial z}[\Xi(y,z)-Z(y,z)]_{y=z=x}, \tag{2.10b}$$

$$V_T(x)=\frac{1}{3x}\left[\frac{2}{x}-\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right]\frac{\partial^2}{\partial y\partial z}[\Xi(y,z)-Z(y,z)]_{y=z=x}. \tag{2.10c}$$

3. THE $\Lambda-\Lambda$ POTENTIAL

We shall use primes to distinguish our quantities from those used in Sec. 2. The potential $V'(|\mathbf{x}_1-\mathbf{x}_2|)$ acting between two Λ particles at $\mathbf{x}_1$ and $\mathbf{x}_2$ can be written

$$V'(x)=V_0'(x)+V_S'(x)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+V_T'(x)S_{12}, \tag{3.1}$$

where $V_0'(x)$, $V_S'(x)$ and $V_T'(x)$ are given by Eqs. (2.10) but with the functions $\Xi(y,z)$, $Z(y,z)$ now replaced by the functions $\Xi'(y,z)$, $Z'(y,z)$ defined as

$$\Xi'(y,z)=3(4\pi)^2\Big[f_\Lambda^4\{F_{\Delta\Delta}(y,z)+2G_{\Delta\Delta}(y,z)\}+\frac{f_\Lambda^2}{6\pi^2}\int_0^\infty\frac{dp}{\omega_p}\sigma_3(p)\{2F_{\Delta\omega_p}(y,z)+G_{\Delta\omega_p}(y,z)\}$$
$$+\frac{1}{144\pi^4}\int_0^\infty\int_0^\infty\frac{dp\,dp'}{\omega_p\omega_{p'}}\sigma_3(p)\sigma_3(p')\{4F_{\omega_p\omega_{p'}}(y,z)+5G_{\omega_p\omega_{p'}}(y,z)\}\Big], \tag{3.2a}$$

$$Z'(y,z)=3(4\pi)^2\Big[f_\Lambda^4F_{\Delta\Delta}(y,z)+\frac{f_\Lambda^2}{6\pi^2}\int_0^\infty\frac{dp}{\omega_p}\sigma_3(p)\{2F_{\Delta\omega_p}(y,z)+3G_{\Delta\omega_p}(y,z)\}$$
$$+\frac{1}{144\pi^4}\int_0^\infty\int_0^\infty\frac{dp\,dp'}{\omega_p\omega_{p'}}\sigma_3(p)\sigma_3(p')\{4F_{\omega_p\omega_{p'}}(y,z)+3G_{\omega_p\omega_{p'}}(y,z)\}\Big]. \tag{3.2b}$$

4. THE RECOIL CORRECTION

The neglect of the recoil kinetic energy of the baryons is least reasonable in the evaluation of the graph in Fig. 1. In the formula for the $N-\Lambda$ potential, part of the function $G_{0\Delta}(y,z)$ arises from this diagram, while the function F arises only from the crossed diagram. The denominator $(0+\Delta)$ in $G_{0\Delta}$ is the energy denominator of the no meson intermediate state of Fig. 1. A rough estimate of the effect of the recoil kinetic energy of the baryons in the intermediate state can be obtained by setting the magnitudes of the momenta of these baryons equal to that of the exchanged pion, namely k. The denominator $(0+\Delta)$ in $G_{0\Delta}$ is then replaced by $(\Delta+(k^2/2m_N^2)+(k^2/2m_\Sigma^2))$ and is included in the integration over k.

Similarly, in the formula for the $\Lambda-\Lambda$ potential, the denominator 2Δ which appears in $G_{\Delta\Delta}(y,z)$ is replaced by $2(\Delta+(k^2/2m_\Sigma^2))$.

5. CALCULATIONS

It is convenient to replace the integrals over $\sigma_3(p)$ and $\sigma_{33}(p)$ in Eqs. (2.6) by a simple rational approximation suggested by Fubini.[17] The integrals over p always have the form

$$\frac{1}{12\pi^2}\int_0^\infty \frac{dp}{\omega_p}\frac{\sigma(p)}{(\omega_k+\omega_p)}. \tag{5.1}$$

The $\pi-N$ and $\pi-\Lambda$ cross sections may both be written in the Breit-Wigner form,

$$\sigma(p)=\frac{12\pi}{p^2}\frac{[\Gamma(p)/2]^2}{[E-E_r]^2+[\Gamma(p)/2]^2}, \tag{5.2}$$

$$\Gamma(p)=\frac{(pa)^3}{1+(pa)^2}\gamma^2. \tag{5.3}$$

Here a is the channel radius which we take to be 1. The width $\Gamma(p)$ at the resonance energy E_r is 90 MeV for $\sigma_{33}(p)$ and 50 MeV for $\sigma_3(p)$. The quantity E_r and the total energy of the pion-baryon system E are both referred to the center-of-mass frame. In evaluating (5.2) we replace $(E-E_r)$ by its approximate value $(\omega_p-\omega_r)$, where $\omega_r=1.91$ for $\sigma_{33}(p)$ and $\omega_r=1.79$ for $\sigma_3(p)$, in units of the pion mass. Equation (5.3) taken at the resonance energy then gives $\gamma^2=38$ MeV for $\sigma_{33}(p)$ and $\gamma^2=25$ MeV for $\sigma_3(p)$. The quantity (5.1) was evaluated numerically for several values of ω_p between 0 and 6 and it was found to be representable to an accuracy of within 0.5% by the form suggested by Fubini,[15] $g_{N,\Lambda}^2/(\omega_k+\omega_{N,\Lambda})$, where

$$g_N^2=0.057,\quad \omega_N=1.83 \quad \text{for} \quad \sigma_{33}(p),$$
$$g_\Lambda^2=0.047,\quad \omega_\Lambda=1.79 \quad \text{for} \quad \sigma_3(p).$$

With these approximations for the integrals over the cross sections, the Eqs. (2.6) and (3.2) can be written

$$\Xi(y,z)+Z(y,z)=6(4\pi)^2[f_N^2f_\Lambda^2\{F_{0\Delta}(y,z)+G_{0\Delta}(y,z)\}+2f_N^2g_\Lambda^2\{F_{0\omega_\Lambda}(y,z)+G_{0\omega_\Lambda}(y,z)\}$$
$$+4f_\Lambda^2g_N^2\{F_{\Delta\omega_N}(y,z)+G_{\Delta\omega_N}(y,z)\}/3+8g_N^2g_\Lambda^2\{F_{\omega_N\omega_\Lambda}(y,z)+G_{\omega_N\omega_\Lambda}(y,z)\}/3], \tag{5.4a}$$

$$\Xi(y,z)-Z(y,z)=6(4\pi)^2[f_N^2f_\Lambda^2G_{0\Delta}(y,z)-f_N^2g_\Lambda^2G_{0\omega_\Lambda}(y,z)-\tfrac{2}{3}f_\Lambda^2g_N^2G_{\Delta\omega_N}(y,z)+\tfrac{2}{3}g_N^2g_\Lambda^2G_{\omega_N\omega_\Lambda}(y,z)], \tag{5.4b}$$

$$\Xi'(y,z)+Z'(y,z)=6(4\pi)^2[f_\Lambda^4\{F_{\Delta\Delta}(y,z)+G_{\Delta\Delta}(y,z)\}+4f_\Lambda^2g_\Lambda^2\{F_{\Delta\omega_\Lambda}(y,z)+G_{\Delta\omega_\Lambda}(y,z)\}$$
$$+4g_\Lambda^4\{F_{\omega_\Lambda\omega_\Lambda}(y,z)+G_{\omega_\Lambda\omega_\Lambda}(y,z)\}], \tag{5.5a}$$

$$\Xi'(y,z)-Z'(y,z)=6(4\pi)^2[f_\Lambda^4G_{\Delta\Delta}(y,z)-2f_\Lambda^2g_\Lambda^2G_{\Delta\omega_\Lambda}(y,z)+g_\Lambda^4G_{\omega_\Lambda\omega_\Lambda}(y,z)]. \tag{5.5b}$$

These quantities and their derivatives were evaluated numerically using an IBM-1620 computer. We took the cutoff momentum k_m to be 6 and the coupling constants f_N^2 and f_Λ^2 to be 0.08.

In the Figs. 3 we present the $N-\Lambda$ and $\Lambda-\Lambda$ potentials as continuous lines. The potentials which would obtain in the absence of the resonances were calculated by setting $g_N^2=g_\Lambda^2=0$ and are given as dashed lines. Both these calculations employ the recoil correction. The dotted lines in Figs. 3 are the potentials which result when the resonances are taken into account but the recoil corrections are omitted.

It is clear that the presence of the resonances leads to a considerable increase in the central spin-independent potential V_0 and to a decrease in the tensor potential V_T. The resonances tend to diminish the magnitude of the spin-dependent central potential V_S, and actually cause it to change sign at short distances. The effect of the recoil correction can be seen to be most pronounced at short distances, where it leads to diminutions in the magnitudes of all potentials. At larger distances the recoil correction becomes negligible. All these features are common to the $N-\Lambda$ and $\Lambda-\Lambda$ potentials.

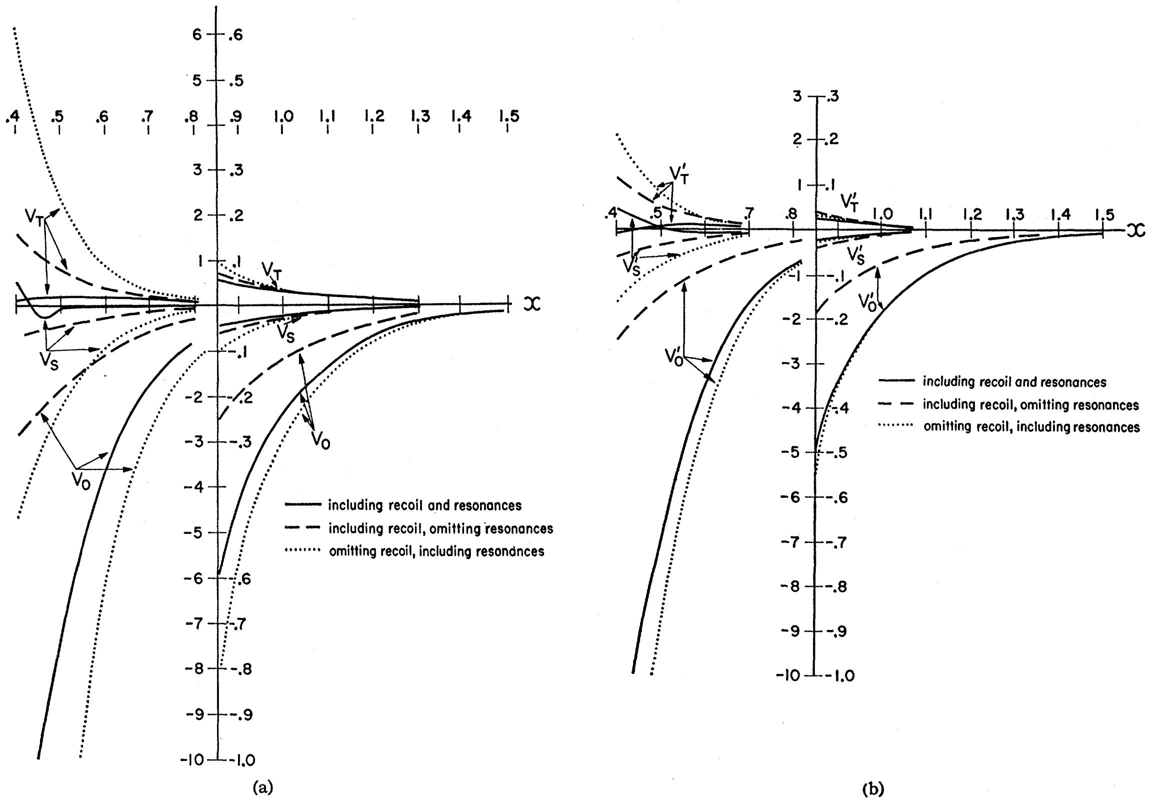
[17] S. Fubini, Nuovo Cimento **3**, 1425 (1956). An alternative approximation is to replact $\sigma(p)$ in Eq. (5.1) by the quantity $12\pi^2g^2p_r\delta(\omega_p-\omega_r)$, where $g^2=\Gamma(p_r)/2p_r^3$. The integral then reduces exactly to the form $g^2/(\omega_k+\omega_r)$. This approximation will be valid when the denominator $\omega_p(\omega_k+\omega_p)$ of Eq. (5.1) varies only slowly across the resonance region of $\sigma(p)$. For $\sigma_{33}(p)$ and $\sigma_3(p)$, this approximation is less accurate then the one adopted.

6. DISCUSSION

The experimental evidence for the $N-\Lambda$ interaction comes from the study of Λ hypernuclei. The evidence for double hypernuclei is as yet meagre, and so our results for the $\Lambda-\Lambda$ interaction will not be discussed in this section. From the analysis of the mesonic decay of ${}_\Lambda H^4$, the $N-\Lambda$ potential appears to be deeper in the singlet state than in the triplet state, although this has not been established conclusively.[1]

In the singlet state the central potential is $V_1=V_0-3V_S$ and in the triplet state it is $V_3=V_0+V_S$. Our calculations for these two quantities are given in Fig. 4 for the $N-\Lambda$ case. When the resonances are included we see that at short distances the potential is slightly more attractive in the singlet state but farther out the triplet potential is stronger. If the resonances are omitted, the triplet state is preferred throughout.

Lichtenberg and Ross[10] have used the Brueckner-Watson method to obtain a result for the two-channel $N-\Lambda$ potential matrix (1.2), taking no account of the 3−3 or Y_1^* resonances. They solved the Schrödinger equation with this potential outside a hard core and



FIGS. 3(a) and 3(b). The $N-\Lambda$ and $\Lambda-\Lambda$ potentials. The quantities V_0, V_S, V_T, V_0', V_S', V_T' of Eqs. (2.9) and (3.1) are plotted against distance x. The calculations which took account of the recoil correction give rise to the continuous lines (where the resonances are included) and to the dashed lines (where the resonances are omitted). The dotted lines result from the calculation which ignored the recoil correction but took account of the resonances. For $x>0.85$, the potentials have been increased by the factor 10.

derived values for the singlet and triplet scattering lengths. The singlet scattering length exceeded the triplet scattering length, implying that the effective one-channel potential would be more attractive for the singlet state than for the triplet state. This result is in disagreement with ours, for we find that in the absence of resonances the potential is more attractive in the triplet state, provided that a common hard core is assumed for the two states. The disagreement may imply that the single-channel treatment is inadequate at short distances, but we consider that our estimates of the potentials at large distances are reliable, as in that region the main source of error in our calculation, which is the recoil correction, is small.

Lichtenberg and Ross chose the coupling constants f_N^2, $f_{\Lambda\Sigma}^2$, $f_{\Sigma\Sigma}^2$ to be all equal. De Swart and Iddings[3] refined the work of Lichtenberg and Ross and varied the core radius and the coupling constants to fit the scattering lengths derived from the potential deduced from the experimental hypernuclear binding energies. The extremely small branching ratio[18] for the decay $Y_1^*\rightarrow\pi+\Sigma$ compared with the decay $Y_1^*\rightarrow\pi+\Lambda$ suggests that $f_{\Sigma\Sigma}^2$ is in fact much smaller than $f_{\Lambda\Sigma}^2$.

[18] R. H. Dalitz, in *Proceedings of the International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 391.

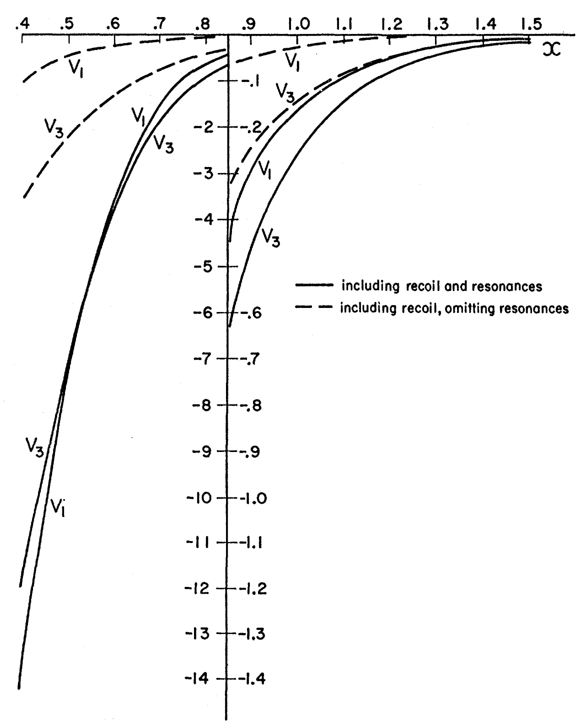


FIG. 4. The $N-\Lambda$ central potential in singlet and triplet states.

In our work, by using Eq. (5.2) for the $\pi-\Lambda$ cross section, we have assumed that the ratio $(Y_1^*\rightarrow\pi+\Sigma)/(Y_1^*\rightarrow\pi+\Lambda)$ is zero. Thus, although our calculation does not involve $f_{\Sigma\Sigma}^2$, we have made an implicit assumption that $f_{\Sigma\Sigma}^2=0$. It would be interesting to repeat the calculation in the two-channel formalism with $f_{\Sigma\Sigma}^2$ set equal to zero and with the effects of the resonances included.

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APPENDIX: THE "MHS" APPROXIMATION AND THE EFFECT OF THE CUTOFF

Matsumoto, Hamada, and Sugawara[14] have suggested that, since the quantities $\omega_k/(\omega_k+\lambda)$ and $\omega_k/(\omega_k+\mu)$ are slowly varying functions of k, the integrals $F_{\lambda\mu}(y,z)$, $G_{\lambda\mu}(y,z)$ may be replaced by the approximations

$$F_{\lambda\mu}(y,z)\cong\frac{a_\lambda(y+z)a_\mu(y+z)}{2(2\pi)^3yz}\int_0^\infty dk\frac{v_k^4k\sin k(y+z)}{\omega_k^3},\quad\text{(A1a)}$$

$$G_{\lambda\mu}(y,z)\cong\frac{a_\lambda(y+z)a_\mu(y+z)}{2(2\pi)^3yz(\lambda+\mu)}\int_0^\infty dk\frac{v_k^4k\sin k(y+z)}{\omega_k^2}.\quad\text{(A1b)}$$

If v_k is set equal to unity, the above integrals may be performed analytically. The function $a_\lambda(y+z)$ is chosen to be the value of $\omega_k/(\omega_k+\lambda)$ at the first maximum of $\sin k(y+z)$, and will then be a slowly varying function of $(y+z)$. If the differentiations in Eqs. (2.10) are carried out treating the functions a_λ as constant, the potentials reduce to the following analytic forms:

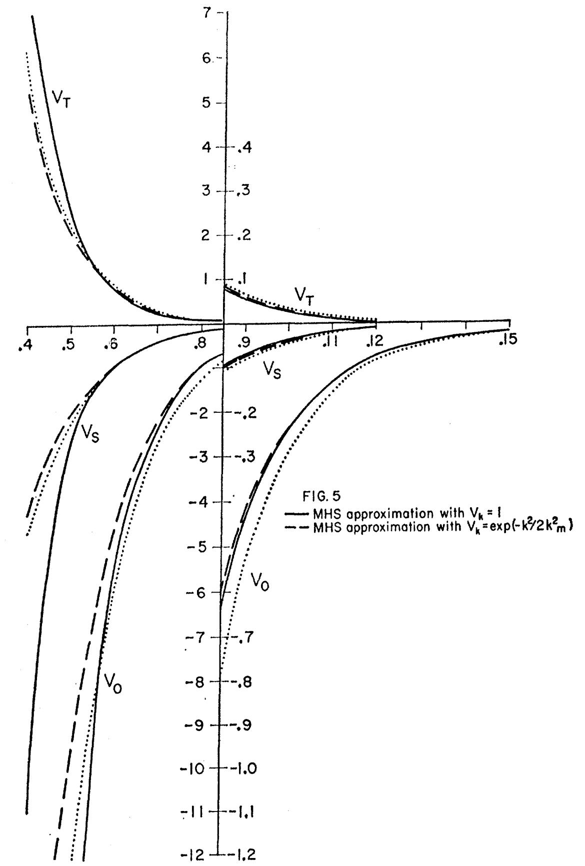

FIG. 5. The validity of the MHS approximation for the $N-\Lambda$ potential. The continuous curves are the results of the MHS approximation with the cutoff factor $v_k=1$. These curves can be obtained analytically by Eqs. (A3). The dashed curves are the results of the MHS approximation with $v_k=\exp(-k^2/2k_m^2)$, and the dotted curves are the results obtained when the MHS approximation is not made. In all cases the recoil has been ignored.

$$V_0(x)=-\frac{3}{\pi}\{f_N^2+4g_N^2a_{\omega N}(2x)/3\}\{f_\Lambda^2a_\Delta(2x)+2g_\Lambda^2a_{\omega_\Lambda}(2x)\}\left\{\left(\frac{23}{x^4}+\frac{12}{x^2}\right)K_1(2x)+\left(\frac{23}{x^3}+\frac{4}{x}\right)K_0(2x)\right\}$$
$$-3\left\{\frac{f_N^2f_\Lambda^2a_\Delta(2x)}{\Delta}+\frac{2f_N^2g_\Lambda^2a_{\omega_\Lambda}(2x)}{\omega_\Lambda}+\frac{4}{3}\frac{f_\Lambda^2g_N^2a_\Delta(2x)a_{\omega N}(2x)}{\Delta+\omega_N}+\frac{8}{3}\frac{g_N^2g_\Lambda^2a_{\omega N}(2x)a_{\omega_\Lambda}(2x)}{\omega_N+\omega_\Lambda}\right\}$$
$$\times\left\{\frac{6}{x^4}+\frac{12}{x^3}+\frac{10}{x^2}+\frac{4}{x}+1\right\}\frac{e^{-2x}}{x^2},\quad\text{(A2a)}$$

$$V_S(x)=-2\left\{\frac{f_N^2f_\Lambda^2a_\Delta(2x)}{\Delta}-\frac{f_N^2g_\Lambda^2a_{\omega_\Lambda}(2x)}{\omega_\Lambda}-\frac{2}{3}\frac{f_\Lambda^2g_N^2a_\Delta(2x)a_{\omega N}(2x)}{\Delta+\omega_N}+\frac{2}{3}\frac{g_N^2g_\Lambda^2a_{\omega N}(2x)a_{\omega_\Lambda}(2x)}{\omega_N+\omega_\Lambda}\right\}$$
$$\times\left\{\frac{3}{x^3}+\frac{6}{x^2}+\frac{5}{x}+2\right\}\frac{e^{-2x}}{x^3},\quad\text{(A2b)}$$

$$V_T(x)=2\left\{\frac{f_N{}^2f_\Lambda{}^2a_\Delta(2x)}{\Delta}-\frac{f_N{}^2g_\Lambda{}^2a_{\omega_\Lambda}(2x)}{\omega_\Lambda}-\frac{2}{3}\frac{f_\Lambda{}^2g_N{}^2a_\Delta(2x)a_{\omega_N}(2x)}{\Delta+\omega_N}+\frac{2}{3}\frac{g_N{}^2g_\Lambda{}^2a_{\omega_N}(2x)a_{\omega_\Lambda}(2x)}{\omega_N+\omega_\Lambda}\right\}$$
$$\times\left\{\frac{3}{x^3}+\frac{6}{x^2}+\frac{4}{x}+1\right\}\frac{e^{-2x}}{x^3},\quad\text{(A2c)}$$

for the $N-\Lambda$ system, and

$$V_0'(x)=-\frac{3}{\pi}\{f_\Lambda{}^2a_\Delta(2x)+2g_\Lambda{}^2a_{\omega_\Lambda}(2x)\}^2\left\{\left(\frac{23}{x^4}+\frac{12}{x^2}\right)K_1(2x)+\left(\frac{23}{x^3}+\frac{4}{x}\right)K_0(2x)\right\}$$
$$-\frac{3}{2}\left\{\frac{f_\Lambda{}^4[a_\Delta(2x)]^2}{\Delta}+\frac{8f_\Lambda{}^2g_\Lambda{}^2a_\Delta(2x)a_{\omega_\Lambda}(2x)}{\Delta+\omega_\Lambda}+\frac{4g_\Lambda{}^4[a_{\omega_\Lambda}(2x)]^2}{\omega_\Lambda}\right\}\left\{\frac{6}{x^4}+\frac{12}{x^3}+\frac{10}{x^2}+\frac{4}{x}+1\right\}\frac{e^{-2x}}{x^2},\quad\text{(A3a)}$$

$$V_S'(x)=-\left\{\frac{f_\Lambda{}^4[a_\Delta(2x)]^2}{\Delta}-\frac{4f_\Lambda{}^2g_\Lambda{}^2a_\Delta(2x)a_{\omega_\Lambda}(2x)}{\Delta+\omega_\Lambda}+\frac{g_\Lambda{}^4[a_{\omega_\Lambda}(2x)]^2}{\omega_\Lambda}\right\}\left\{\frac{3}{x^3}+\frac{6}{x^2}+\frac{5}{x}+2\right\}\frac{e^{-2x}}{x^3},\quad\text{(A3b)}$$

$$V_T'(x)=\left\{\frac{f_\Lambda{}^4[a_\Delta(2x)]^2}{\Delta}-\frac{4f_\Lambda{}^2g_\Lambda{}^2a_\Delta(2x)a_{\omega_\Lambda}(2x)}{\Delta+\omega_\Lambda}+\frac{g_\Lambda{}^4[a_{\omega_\Lambda}(2x)]^2}{\omega_\Lambda}\right\}\left\{\frac{3}{x^3}+\frac{6}{x^2}+\frac{4}{x}+1\right\}\frac{e^{-2x}}{x^3},\quad\text{(A3c)}$$

for the $\Lambda-\Lambda$ system, where K_n is the modified Bessel function.

The effect of the presence of the resonance terms, which are the ones which involve $g_N{}^2$ and $g_\Lambda{}^2$, can now be clearly seen.

This "MHS" approximation leads to a useful simplification only when the cutoff factor v_k is set equal to unity. The divergences in the derivatives of the functions $F_{\lambda\mu}(y,z)$, $G_{\lambda\mu}(y,z)$ are avoided by differentiating after the integration over k has been performed. It is interesting to see the variations in the final potentials caused by the replacement of $v_k=\exp(-k^2/2k_m{}^2)$ by $v_k=1$, and to this end we calculated numerically the integrals in (A1) with $v_k=\exp(-k^2/2k_m{}^2)$, and evaluated the corresponding potentials. These are compared with the potentials given by (A2) and (A3) in Fig. 5 for the $N-\Lambda$ case. For comparison we also present the potentials calculated without recoil and without any approximation to F or G. The effect of recoil cannot be included in an MHS approximation. It can be seen that the MHS approximation is useful for radii exceeding 0.6 units, but that at shorter distances the neglect of cutoff in the MHS approximation produces overestimates of the more acceptable potential in which the cutoff has been included. At short distances of course, the recoil correction also becomes very important, and it may be necessary to work with the two-channel formalism to include the recoil satisfactorily.